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Representations for the Drazin Inverse of a Modified Matrix

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Abstract. In this paper expressions for the Drazin inverse of a modified matrix $A - CD^dB$ are presented in terms of the Drazin inverses of A and the generalized Schur complement $D - BA^dC$ under weaker restrictions. Our results generalize and unify several results in the literature and the Sherman-Morrison-Woodbury formula.

1. Introduction

The classical Sherman-Morrison-Woodbury formula reads

$$(A - CD^{-1}B)^{-1} = A^{-1} + A^{-1}C(D - BA^{-1}C)^{-1}BA^{-1},$$

where A and D are invertible matrices (not necessarily with the same size) and B and C are matrices with the appropriate size such that $D - BA^{-1}C$ (and so $A - CD^{-1}B$) is invertible ([25, 30]). The matrix $A - CD^{-1}B$ is called, especially in the case where D is the identity matrix, a modified matrix of A, and $D - BA^{-1}C$ is called the Schur complement. The Sherman-Morrison-Woodbury formula allows one to compute the inverse of a modified matrix in terms of the inverses of the original matrix and its Schur complement. Inverse matrix modification formulae of such type have been studied extensively and has numerous applications in various fields such as statistics, networks, structural analysis, numerical analysis, optimization and partial different equations, etc., see [13, 14, 17]. Formulae of such type have been developed in the context of generalized inverses, such as the Moore-Penrose inverse [1, 20], the weighted Moore-Penrose inverse [28], the group inverse [6], the weighted Drazin inverse [8], the generalized Drazin inverse [11, 19], and especially the Drazin inverse [10, 12, 23, 24, 29].

In this paper, we are interested in the inverse matrix modification formula in the setting of the Drazin inverse.

The Drazin inverse of a complex square matrix A is the unique matrix A^d such that

$$AA^d = A^dA, \quad A^dAA^d = A^d, \quad A^k = A^{k+1}A^d,$$

where k is the smallest non-negative integer such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$, called index of A and denoted by $\operatorname{ind}(A)$. If $\operatorname{ind}(A) = 1$, then A^d is called the group inverse of A and denoted by $A^\#$. The Drazin inverse is

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a generalization of inverses and group inverses of matrices. There are widespread applications of Drazin inverses in various fields, such as differential equations, control theory, Markov chains, iterative methods and so on (see [2, 4]).

Wei [29] derived explicit expressions of the Drazin inverse of a modified matrix $A - CB$ under certain circumstances, which extends results of [22, 26] and can be used to present a perturbation bound for the Drazin inverse studied by Campbell and Meyer [3]. Recently, Cvetković-Ilić and Ljubisavljević [10] and Dopazo and Martínez-Serrano [12] extended results of Wei [29] to the modified matrix $A - CD^dB$. Mosić [23] and Shakoor, Yang and Ali [24] generalized results of [10, 12]. These results are useful for perturbation problems and updating finite Markov chains.

In this paper, based on an observation on Dedekind finiteness of unital matrix subalgebras, we relax and remove some restrictions in theorems in [10, 12, 23, 24, 29] and give representations of $(A - CD^dB)^d$ under fewer and weaker conditions. Our results generalize and unify results of these literatures and the Sherman-Morrison-Woodbury formula. We also derive a new formula for the Drazin inverse of $A - CD^dB$, a corollary of which recovers a generalization of Jacobson's Lemma (see [5, Theorem 3.6]) for the case of matrices.

Throughout this paper, let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices, $A \in \mathbb{C}^{n\times n}$, $D \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times m}$. Let I denote the identity matrix of proper size. Conventionally, write A^π for $I - AA^d$. For simplicity, we will write S for the modified matrix $A - CD^dB$ and Z for the generalized Schur complement $D - BA^dC$.

We will adopt the conventions that $A^0 = I$ for any square matrix A and $\sum_{i=0}^{k} * = 0$ in case $k < 0$.

2. Drazin Inverse of a Modified Matrix

We start with a well-known result.

Lemma 2.1. *([16, Theorem 2.1]) Let P and Q be $n \times n$ matrices. If $PQ = 0$, then*

$$(P + Q)^d = Q^\pi \sum_{i=0}^{t-1} Q^i (P^d)^{i+1} + \sum_{i=0}^{s-1} (Q^d)^{i+1} P^i P^\pi,$$

where $s = \operatorname{ind}(P)$ and $t = \operatorname{ind}(Q)$.

In what follows, let $S_A = AA^dSAA^d$.

Lemma 2.2. *If $A^\pi CD^dB = 0$, then $S_A = SAA^d$ and*

$$S^d = S_A^d + \sum_{i=0}^{k-1} (S_A^d)^{i+2} SA^i A^\pi,$$

where $k = \operatorname{ind}(A)$.

Proof. Since $A^\pi CD^dB = 0$, we first note that $S_A = AA^dSAA^d = SAA^d$ and $S = SA^\pi + S_A$. Then $A^\pi S_A = 0$ and

$$(SA^\pi)^i = S(A^\pi S)^{i-1}A^\pi = S(AA^\pi)^{i-1}A^\pi = SA^{i-1}A^\pi$$

for any positive integer i. Let $k = \operatorname{ind}(A)$. It has been known that k is the least nonnegative integer such that $A^kA^\pi = 0$. Thus SA^π is nilpotent and $k \le \operatorname{ind}(SA^\pi) \le k + 1$, and so $(SA^\pi)^d = 0$ and $(SA^\pi)^\pi = I$. Let $s = \operatorname{ind}(SA^\pi)$. Then Lemma 2.1 implies that

$$S^d = \sum_{i=0}^{s} (S_A^d)^{i+1} (SA^\pi)^i = S_A^d + \sum_{i=0}^{s-1} (S_A^d)^{i+2} SA^i A^\pi.$$

Since $s-1 \leq k \leq s$ and $A^iA^\pi = 0$ for any $i \geq k$, we have

$$S^d = S_A^d + \sum_{i=0}^{k-1} (S_A^d)^{i+2} SA^iA^\pi,$$

as desired. □

Lemma 2.3. *Let X, Y and e be $n \times n$ matrices and $e^2 = e$. If $XeY = e$, then $eYeXe = e$.*

Proof. Let $W = \{M \in \mathbb{C}^{n\times n} \mid eM = Me = M\}$. Then W is a finite dimensional algebra over $\mathbb{C}$ with identity e, and so W is Dedekind finite (see [18, Corollary 21.27]). Note that eXe, $eYe \in W$ and $(eXe)(eYe) = e$. Then $(eYe)(eXe) = e$, that is, $eYeXe = e$. □

If $\mathrm{ind}(A) = 1$, then A^d is called the group inverse of A and denoted by $A^\#$. In what follows, let $H = BA^d$ and $K = A^dC$.

Lemma 2.4. *Let $S_A = AA^dSAA^d$ and $M = A^d + KZ^dH$. Then the following statements are equivalent:*

(1) $KD^\pi Z^dH = KD^dZ^\pi H$;
(2) $S_AM = AA^d$;
(3) $MS_A = AA^d$;
(4) $KZ^\pi D^dH = KZ^dD^\pi H$.

Furthermore, if one of (1)–(4) holds, then S_A has the group inverse

$$S_A^\# = A^d + KZ^dH.$$

Proof. Let $A^e = AA^d$ and $Z^e = ZZ^d$. Then

$$S_AM = A^e + AKZ^dH - AKD^dH - AKD^d(D-Z)Z^dH.$$

Thus (2) holds if and only if $AK(Z^d - D^d - D^d(D-Z)Z^d)H = 0$, or equivalently $K(D^\pi Z^d - D^dZ^\pi)H = 0$, that is, (1) holds. Similarly, (3) is equivalent to (4). Lemma 2.3 implies equivalence of (2) and (3). Furthermore, (2) and (3) give $S_A^\# = M = A^d + KZ^dH$. □

Now we can give our first main result.

Theorem 2.5. *If $A^\pi CD^dB = 0$ and $KD^\pi Z^dH = KD^dZ^\pi H$, then*

$$S^d = A^d + KZ^dH + \sum_{i=0}^{k-1} (A^d + KZ^dH)^{i+2} SA^iA^\pi$$

with $H = BA^d$ and $K = A^dC$, or alternatively

$$\begin{aligned} S^d = {} & A^d + A^dCZ^dBA^d - \sum_{i=0}^{k-1} (A^d + A^dCZ^dBA^d)^{i+1} A^dCZ^dBA^iA^\pi \\ & + \sum_{i=0}^{k-1} (A^d + A^dCZ^dBA^d)^{i+1} A^dC(Z^dD^\pi - Z^\pi D^d)BA^i, \end{aligned}$$

where $k = \mathrm{ind}(A)$.

Proof. Since $A^\pi CD^dB = 0$, we have $S_A = AA^dSAA^d = SAA^d$. Thus the first equation follows from Lemma 2.2 and Lemma 2.4. Note that

$$\begin{aligned}(A^d + KZ^dH)SA^\pi =&(AA^d - KD^dB + KZ^dBAA^d - KZ^d(D - Z)D^dB)A^\pi\\ =& - KZ^\pi D^dBA^\pi - KZ^d(I - D^\pi)BA^\pi,\\ =&K(Z^dD^\pi - Z^\pi D^d)BA^\pi - KZ^dBA^\pi,\end{aligned}$$

Since $KD^\pi Z^dH = KD^dZ^\pi H$, by Lemma 2.4 we have

$$K(Z^dD^\pi - Z^\pi D^d)BA^\pi = K(Z^dD^\pi - Z^\pi D^d)B - K(Z^dD^\pi - Z^\pi D^d)HA = K(Z^dD^\pi - Z^\pi D^d)B,$$

whence the second equation follows from the first one. □

Remark 2.6. *From the second expression of S^d in Theorem 2.5, we can more clearly see several results in the literature how to be generalized. In addition, observing that*

$$(A^d + KZ^dH)SA^\pi = K(Z^dD^\pi - Z^\pi D^d)BA^\pi - KZ^dBA^\pi = -K(Z^dD + Z^\pi)D^dBA^\pi,$$

we can get another expression of S^d:

$$S^d = A^d + A^dCZ^dBA^d - \sum_{i=0}^{k-1}(A^d + A^dCZ^dBA^d)^{i+1}A^dC(Z^dD + Z^\pi)D^dBA^iA^\pi,$$

where $k = \mathrm{ind}(A)$.

The following result is a fairly direct consequence of Theorem 2.5.

Corollary 2.7. *If $A^\pi CD^dB = 0$, $CD^\pi Z^dB = 0$ and $CD^dZ^\pi B = 0$, then*

$$\begin{aligned}S^d =& A^d + A^dCZ^dBA^d - \sum_{i=0}^{k-1}(A^d + A^dCZ^dBA^d)^{i+1}A^dCZ^dBA^iA^\pi\\ &+ \sum_{i=0}^{k-1}(A^d + A^dCZ^dBA^d)^{i+1}A^dC(Z^dD^\pi - Z^\pi D^d)BA^i,\end{aligned}$$

where $k = \mathrm{ind}(A)$.

We now analyse some special cases of the preceding theorems, some of which give and generalize the Sherman-Morrison-Woodbury formula and results of [10, 12, 23, 24, 29].

It is easy to verify that $D^\pi Z^d = 0$ and $D^dZ^\pi = 0$ if and only if $D^\pi = Z^\pi$, that is, D and Z have the same eigenprojection at zero. Matrices with equal eigenprojections at zero were studied in [7] to give error bounds for the Drazin inverse of a perturbation. So the following consequence of Theorem 2.5 is of independent interest.

Corollary 2.8. *If $A^\pi CD^dB = 0$ and $D^\pi = Z^\pi$, then*

$$S^d = A^d + A^dCZ^dBA^d - \sum_{i=0}^{k-1}(A^d + A^dCZ^dBA^d)^{i+1}A^dCZ^dBA^iA^\pi,$$

where $k = \mathrm{ind}(A)$.

In [10, 12, 24], expressions of the Drazin inverse of $A - CD^dB$ are given under the following conditions

(1) $A^\pi C = 0,\ BA^\pi = 0,\ CD^\pi Z^dB = 0,\ CD^dZ^\pi B = 0,\ CZ^dD^\pi B = 0$ and $CZ^\pi D^dB = 0$;

(2) $A^\pi C = 0,\ CD^\pi = 0,\ Z^\pi B = 0,\ D^\pi B = 0$ and $CZ^\pi = 0$;
(3) $A^\pi C = 0,\ CD^\pi Z^d B = 0,\ CD^d Z^\pi B = 0,\ CZ^d D^\pi B = 0$ and $CZ^\pi D^d B = 0$.

Corollary 2.7 relaxes the first condition and drops the last two ones in each item of (1)-(3) and gives a unified generalization of [10, Theorem 2.1], [12, Theorem 2.1] and [24, Theorem 2.1].

Corollary 2.7 also generalizes Theorem 1 and Theorem 2 of [23], in which Mosić made the following assumptions:

(4) $A^\pi C = CD^\pi,\ D^\pi B = 0$ and $DZ^\pi = 0$;
(5) $A^\pi C = CD^\pi,\ D^\pi B = 0,\ Z^\pi B = 0$ and $CZ^\pi = 0$.

Indeed, observe that $A^\pi C = CD^\pi$ implies $A^\pi CD^d B = 0$ and $BA^d CD^\pi = 0$, or equivalently $(D - Z)D^\pi = 0$, while $D^\pi B = 0$ implies $D^\pi BA^d C = 0$, or equivalently $D^\pi (D - Z) = 0$. Then we have $ZD^\pi = D^\pi Z$, and particularly $Z^d D^\pi = D^\pi Z^d$. Now it is easy to see that either of (4) and (5) implies the conditions of Corollary 2.7.

If we suppose D is the identity matrix, then Corollary 2.7 gives a generalization of [29, Theorem 2.1].

If A, D and Z are invertible in Corollary 2.7, then $A^\pi = 0$, $D^\pi = 0$, $Z^\pi = 0$ and ind(A)=0. In this case it is well known that S is also invertible, and so Corollary 2.7 is exactly the Sherman-Morrison-Woodbury formula.

The following theorem, which is a dual version of Theorem 2.5, can be proved similarly.

Theorem 2.9. *If $CD^d BA^\pi = 0$ and $CZ^\pi D^d B = CZ^d D^\pi B$, then*

$$\begin{aligned} S^d = {} & A^d + A^d CZ^d BA^d - \sum_{i=0}^{k-1} A^\pi A^i CZ^d BA^d (A^d + A^d CZ^d BA^d)^{i+1} \\ & + \sum_{i=0}^{k-1} A^i C(D^\pi Z^d - D^d Z^\pi) BA^d (A^d + A^d CZ^d BA^d)^{i+1}, \end{aligned}$$

where $k = \operatorname{ind}(A)$.

Theorem 2.9 generalizes [10, Theorem 2.1], [23, Theorem 3], [12, Theorem 2.2], [24, Theorem 2.2] and [29, Theorem 2.1].

We now turn to the next main result of this section. Let $H = BA^d$, $K = A^d C$, $\Gamma = HK$ and $E = AA^d - K\Gamma^d H$, the first three of which were introduced by Wei [27] to give representations for the Drazin inverses of block matrices.

The following lemma is an immediate consequence of [9, Corollary 3.2].

Lemma 2.10. *Let P, Q and R be $n \times n$ matrices such that $PQ = QP = QR = RP = R^2 = 0$. If Q is nilpotent, then*

$$(P + Q + R)^d = P^d + \sum_{i=0}^{t-1} (P^d)^{i+2} RQ^i,$$

where $t = \operatorname{ind}(Q)$.

Lemma 2.11. *If $A^\pi CD^d B = 0$ and $K\Gamma^d HSA^d = 0$, then*

$$S^d = (ESE)^d + \sum_{i=0}^{t-1} ((ESE)^d)^{i+2} (SE^\pi)^{i+1},$$

where $H = BA^d$, $K = A^d C$, $\Gamma = HK$, $E = AA^d - K\Gamma^d H$ and $t = \operatorname{ind}(E^\pi SE^\pi)$.

Proof. We first note that E is idempotent. Since $A^\pi CD^dB = 0$, we have $A^\pi SE = A^\pi AE = 0$. Thus

$$E^\pi SE = (A^\pi + K\Gamma^d H)SE = K\Gamma^d HSE = 0$$

and

$$E^\pi SE^\pi = E^\pi S = A^\pi S + K\Gamma^d HS = AA^\pi + K\Gamma^d HS.$$

Since $A^\pi K = 0$, $(K\Gamma^d HS)^2 = 0$ and AA^π is nilpotent, we see that $E^\pi SE^\pi$ is nilpotent. Now we use Lemma 2.10 to $S = ESE + E^\pi SE^\pi + ESE^\pi$ to obtain

$$\begin{aligned} S^d &= (ESE)^d + \sum_{i=0}^{t-1}((ESE)^d)^{i+2} SE^\pi (E^\pi SE^\pi)^i \\ &= (ESE)^d + \sum_{i=0}^{t-1}((ESE)^d)^{i+2}(SE^\pi)^{i+1}, \end{aligned}$$

where $t = \mathrm{ind}(E^\pi SE^\pi)$. □

Lemma 2.12. *If $K\Gamma^d HSA^d = 0$ and $K\Gamma^\pi D^d H = 0$, then there exists the group inverse of ESE and $(ESE)^\# = EA^dE$, where $H = BA^d$, $K = A^dC$, $\Gamma = HK$, $E = AA^d - K\Gamma^d H$.*

Proof. We first note that E is idempotent and $E = A^d(A - C\Gamma^d H)$. A calculation gives

$$EK = K - K\Gamma^d\Gamma = K\Gamma^\pi. \tag{1}$$

Then

$$EA^dESE = EA^dSE - EA^dK\Gamma^d HSE = E(AA^d - KD^dB)E = E - EKD^dBE = E - K\Gamma^\pi D^dBE = E.$$

It follows from Lemma 2.3 that $ESEA^dE = E$. Thus ESE has the group inverse EA^dE. □

Theorem 2.13. *If $A^\pi CD^dB = 0$, $K\Gamma^d HSA^d = 0$ and $K\Gamma^\pi D^d H = 0$, then*

$$S^d = (I - K\Gamma^d H)A^d(I - K\Gamma^d H) - \sum_{i=0}^{k-1}((I - K\Gamma^d H)A^d)^{i+2} K\Gamma^d HSA^i - \sum_{i=0}^{k-1}((I - K\Gamma^d H)A^d)^{i+1} K\Gamma^\pi D^d BA^i,$$

where $H = BA^d$, $K = A^dC$, $\Gamma = HK$ and $k = \mathrm{ind}(A)$.

Proof. Let $E = AA^d - K\Gamma^d H$. By Lemma 2.11 and Lemma 2.12 we have

$$\begin{aligned} S^d &= EA^dE + \sum_{i=0}^{t-1}(EA^dE)^{i+2}(SE^\pi)^{i+1} \\ &= EA^dE + \sum_{i=0}^{t-1}(EA^dE)^{i+1}(EA^dESE^\pi)(E^\pi SE^\pi)^i, \end{aligned} \tag{2}$$

where $t = \mathrm{ind}(E^\pi SE^\pi)$. Note that $EA^dESE^\pi = EA^dESA^\pi + EA^dESK\Gamma^d H$. Since $K\Gamma^d HSA^d = 0$, we have $K\Gamma^d HSA^\pi = K\Gamma^d HS$. It follows from (1) and $K\Gamma^\pi D^d H = 0$ that

$$\begin{aligned} EA^dESA^\pi &= EA^d(AA^dSA^\pi - K\Gamma^d HSA^\pi) \\ &= EA^d(-CD^dBA^\pi - K\Gamma^d HS) \\ &= -EKD^dBA^\pi - EA^dK\Gamma^d HS \\ &= -K\Gamma^\pi D^dB - EA^dK\Gamma^d HS. \end{aligned}$$

We first note that $EK\Gamma^d H = 0$. By $K\Gamma^d HSA^d = 0$ again we have $K\Gamma^d HSK = 0$. It follows from $K\Gamma^\pi D^d H = 0$ that

$$\begin{aligned} EA^d ESK\Gamma^d H &= EA^d(AA^d - K\Gamma^d H)SK\Gamma^d H \\ &= EA^d SK\Gamma^d H \\ &= EA^d(A - CD^d B)K\Gamma^d H \\ &= EK\Gamma^d H - EKD^d HC\Gamma^d H \\ &= -K\Gamma^\pi D^d HC\Gamma^d H \\ &= 0. \end{aligned}$$

Then $EA^d ESE^\pi = -K\Gamma^\pi D^d B - EA^d K\Gamma^d HS$. Note that $E^\pi SE^\pi = AA^\pi + K\Gamma^d HS$. Since $A^\pi K = 0$ and $(K\Gamma^d HS)^2 = 0$, we have

$$(E^\pi SE^\pi)^i = (A^\pi A + K\Gamma^d HS)A^{i-1},$$

for any positive integer i, whence $\mathrm{ind}(A) \le \mathrm{ind}(E^\pi SE^\pi) \le \mathrm{ind}(A) + 1$. It follows that

$$\begin{aligned} EA^d ESE^\pi(E^\pi SE^\pi)^i &= (-K\Gamma^\pi D^d B - EA^d K\Gamma^d HS)(A^\pi A + K\Gamma^d HS)A^{i-1} \\ &= -K\Gamma^\pi D^d BA^i - EA^d K\Gamma^d HSA^i, \end{aligned} \tag{3}$$

for any positive integer i. Combining (2) and (3) yields

$$S^d = EA^d E - \sum_{i=0}^{t-1}(EA^d)^{i+2}K\Gamma^d HSA^i - \sum_{i=0}^{t-1}(EA^d)^{i+1}K\Gamma^\pi D^d BA^i.$$

Since $t - 1 \le k \le t$ and $K\Gamma^d HSA^i = K\Gamma^\pi D^d BA^i = 0$ for any $i \ge k$, we have

$$S^d = EA^d E - \sum_{i=0}^{k-1}(EA^d)^{i+2}K\Gamma^d HSA^i - \sum_{i=0}^{k-1}(EA^d)^{i+1}K\Gamma^\pi D^d BA^i.$$

Now the conclusion follows from the facts that $EA^d = (I - K\Gamma^d H)A^d$ and $A^d E = A^d(I - K\Gamma^d H)$. □

Theorem 2.14. *If $A^\pi CD^d B = 0$, $C\Gamma^d ZD^d B = 0$, $C\Gamma^d D^\pi B = 0$ and $C\Gamma^\pi D^d B = 0$, then*

$$S^d = (I - K\Gamma^d H)A^d(I - K\Gamma^d H) + \sum_{i=0}^{k-1}((I - K\Gamma^d H)A^d)^{i+2}K\Gamma^d BA^i A^\pi,$$

where $k = \mathrm{ind}(A)$.

Proof. A calculation yields

$$HS = BA^d A - BA^d CD^d B = BA^d A - (D - Z)D^d B = BA^d A - DD^d B + ZD^d B = -BA^\pi + D^\pi B + ZD^d B.$$

Now it is easy to verify that the conditions of Theorem 2.13 are satisfied and $K\Gamma^d HS = -K\Gamma^d BA^\pi$ and $K\Gamma^\pi D^d B = 0$. Thus the expression of S^d in Theorem 2.13 can be reduced to the desired one. □

Theorem 2.14 generalizes [29, Theorem 2.2], [10, Theorem 2.2] and [24, Theorem 2.8], where the following assumptions are made, respectively,

(1) $A^\pi C = 0$, $Z = 0$, $D = I$, $BA^\pi = 0$ and $C\Gamma^\pi B = 0$;
(2) $A^\pi C = 0$, $Z = 0$, $BA^\pi = 0$, $CD^d\Gamma^\pi B = 0$, $C\Gamma^\pi D^d B = 0$, $CD^\pi\Gamma^d B = 0$, and $C\Gamma^d D^\pi B = 0$;
(3) $A^\pi C = 0$, $Z = 0$, $CD^d\Gamma^\pi B = 0$, $C\Gamma^\pi D^d B = 0$, $CD^\pi\Gamma^d B = 0$ and $C\Gamma^d D^\pi B = 0$.

Theorem 2.14 relaxes and drops some conditions above. Moreover, it is not difficult to give an example that satisfies the conditions of Theorem 2.14 and $Z \neq 0$:

$$A = 1,\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The following theorem may be proved in the same way as Theorem 2.13.

Theorem 2.15. *If $CD^dBA^\pi = 0$, $KD^dZ\Gamma^dH = 0$, $KD^\pi\Gamma^dH = 0$ and $KD^d\Gamma^\pi H = 0$, then*

$$S^d = (I - K\Gamma^dH)A^d(I - K\Gamma^dH) - \sum_{i=0}^{k-1} A^iSK\Gamma^dH(A^d(I - K\Gamma^dH))^{i+2} - \sum_{i=0}^{k-1} A^iCD^d\Gamma^\pi H(A^d(I - K\Gamma^dH))^{i+1},$$

where $H = BA^d$, $K = A^dC$, $\Gamma = HK$ and $k = \mathrm{ind}(A)$.

Theorem 2.15 generalizes [29, Theorem 2.2] [10, Theorem 2.2] and [24, Theorem 2.9].

3. Generalized Jacobson's Lemma

In this section we will present new expressions for $(A - CD^dB)^d$.

Lemma 3.1. *([15] and [21]) Let $M = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$ and $N = \begin{bmatrix} D & 0 \\ C & A \end{bmatrix} \in \mathbb{C}^{n\times n}$, where A and D are square matrices. Then*

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} \text{ and } N^d = \begin{bmatrix} D^d & 0 \\ X & A^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{s-1} (A^d)^{i+2}CD^iD^\pi + A^\pi \sum_{i=0}^{r-1} A^iC(D^d)^{i+2} - A^dCD^d,$$

$r = \mathrm{ind}(A)$ and $s = \mathrm{ind}(D)$.

Lemma 3.2. *If $D^dBAA^d = D^dDBA^d$, then*

$$S_A^d = A^d + A^dCZ_D^dD^dBAA^d - \sum_{i=0}^{s-1} (A^d)^{i+2}CDD^dZ_D^iZ_D^\pi D^dBAA^d,$$

where $S_A = AA^dSAA^d$, $Z_D = DD^dZDD^d$ and $s = \mathrm{ind}(Z_D)$.

Proof. By abuse of notation we write A^{2d} instead of $(A^d)^2$ for any $n \times n$ matrix A. Let $A^e = AA^d$ and $D^e = DD^d$. Note that

$$\begin{bmatrix} S_A & A^eCD^e \\ 0 & DD^e \end{bmatrix} = \begin{bmatrix} AA^e & A^eCD^e \\ D^eBA^e & DD^e \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^dBA^e & I \end{bmatrix}.$$

For short let us introduce the temporary notation

$$M = \begin{bmatrix} AA^e & A^eCD^e \\ D^eBA^e & DD^e \end{bmatrix} \text{ and } N = \begin{bmatrix} I & 0 \\ -D^dBA^e & I \end{bmatrix}.$$

Then Cline's formula gives

$$\begin{bmatrix} S_A & A^eCD^e \\ 0 & DD^e \end{bmatrix}^d = M(NM)^{2d}N.$$

A calculation yields $NM = \begin{bmatrix} AA^e & A^eCD^e \\ D^eBA^e - D^dBA^eA & DD^e - D^dBA^eCD^e \end{bmatrix}$. Since $D^dBAA^d = D^dDBA^d$, we have $D^eBA^e - D^dBA^eA = 0$ and $DD^e - D^dBA^eCD^e = Z_D$. Thus $NM = \begin{bmatrix} AA^e & A^eCD^e \\ 0 & Z_D \end{bmatrix}$. By Lemma 3.1 we have

$$(NM)^{2d} = \begin{bmatrix} A^d & X \\ 0 & Z_D^d \end{bmatrix}^2 = \begin{bmatrix} A^{2d} & A^dX + XZ_D^d \\ 0 & Z_D^{2d} \end{bmatrix},$$

where $X = \sum_{i=0}^{s-1}(A^d)^{i+2}CD^eZ_D^iZ_D^\pi - A^dCZ_D^d$ and $s = \mathrm{ind}(Z_D)$. Note that $XZ_D^d = -A^dCZ_D^{2d}$. Then

$$\begin{bmatrix} S_A & A^eCD^e \\ 0 & DD^e \end{bmatrix}^d = \begin{bmatrix} A^d - XD^dBA^e & X \\ D^eBA^{2d} - YD^dBA^e & Y \end{bmatrix},$$

where $Y = D^eB(A^dX - A^dCZ_D^{2d}) + DZ_D^{2d}$. Hence

$$S_A^d = A^d + A^dCZ_D^dD^dBA^e - \sum_{i=0}^{s-1}(A^d)^{i+2}CD^eZ_D^iZ_D^\pi D^dBA^e,$$

where $s = \mathrm{ind}(Z_D)$. □

Combining Lemma 2.2 and Lemma 3.2 gives the following result.

Lemma 3.3. *If $A^\pi CD^dB = 0$ and $D^dBAA^d = D^dDBA^d$, then*

$$S^d = S_A^d + \sum_{i=0}^{k-1}(S_A^d)^{i+2}SA^iA^\pi,$$

where

$$S_A^d = A^d + A^dCZ_D^dD^dBAA^d - \sum_{i=0}^{s-1}(A^d)^{i+2}CDD^dZ_D^iZ_D^\pi D^dBAA^d,$$

$S_A = AA^dSAA^d,\ Z_D = DD^dZDD^d$, $k = \mathrm{ind}(A)$ *and* $s = \mathrm{ind}(Z_D)$.

To represent S^d in terms of Z^d we need an extra assumption.

Theorem 3.4. *If $A^\pi CD^dB = 0,\ D^\pi BA^dC = 0$ and $D^dBAA^d = D^dDBA^d$, then*

$$S^d = S_A^d + \sum_{i=0}^{k-1}(S_A^d)^{i+2}SA^iA^\pi,$$

where $k = \mathrm{ind}(A),\ S_A = AA^dSAA^d$,

$$S_A^d = A^d + A^dCZ^dD^dBAA^d - \sum_{i=0}^{s-1}(A^d)^{i+2}CDD^dZ^iZ^\pi D^dBAA^d,$$

and $s = \mathrm{ind}(Z)$.

Proof. Let $D^e = DD^d$ and $Z_D = D^eZD^e$. Using an analogous strategy as Lemma 2.2 we get

$$Z^d = Z_D^d + \sum_{i=0}^{t-1}(Z_D^d)^{i+2}ZD^iD^\pi,$$

where $t = \mathrm{ind}(D)$. Note that $Z^dD^e = Z_D^d$, $D^eZ^d = Z^d$ and $Z_D = ZD^e$. Then $Z_D^i = Z^iD^e$ and $Z_D^\pi D^d = (I - Z_DZ^d)D^d = Z^\pi D^d$, implying

$$Z_D^iZ_D^\pi D^d = Z^iD^e(I - Z^e)D^d = Z^iZ^\pi D^d.$$

Now the theorem follows from Lemma 3.3. □

Corollary 3.5. *Let A and D be invertible. If $DB = BA$, then*

$$(A - CD^{-1}B)^d = A^{-1} + A^{-1}CZ^dD^{-1}B - \sum_{i=0}^{s-1} A^{-i-2}CZ^iZ^\pi D^{-1}B,$$

where $s = \mathrm{ind}(Z)$.

If A and D are identity matrices in the corollary above, then we recover a generalization of Jacobson's Lemma (see [5, Theorem 3.6]) for the case of matrices.

Corollary 3.6. *Let $s = \mathrm{ind}(I - BC)$. Then*

$$(I - CB)^d = I + C(I - BC)^dB - \sum_{i=0}^{s-1} C(I - BC)^i(I - BC)^\pi B.$$

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